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Convergence of Cascade Algorithms in Sobolev Spaces Associated with Inhomogeneous Refinement Equations[1]

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In this paper, we consider multivariate inhomogeneous refinement equations of the form $\varphi(x)=\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\,\varphi(2x-\alpha)+g(x)$, $x\in\mathbb{R}^s$, where $\varphi=(\varphi_1,...,\varphi_r)^T$ is the unknown, $g=(g_1,...,g_r)^T$ is a given vector of functions on $\mathbb{R}^s$, and a is a finitely supported refinement mask such that each $a(\alpha)$ is an $r\times r$ (complex) matrix. Let φ_0 be an initial vector of functions in the Sobolev space $W_2^k(\mathbb{R}^s)$. The corresponding cascade algorithm is given by $\varphi_n(x)=\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\,\varphi_{n-1}(2x-\alpha)+g(x)$, $x\in\mathbb{R}^s$, $n=1, 2, ...$. A characterization is given for the strong convergence of the cascade algorithm in the Sobolev space $W_2^k(\mathbb{R}^s)$ $(k\in\mathbb{N})$ in terms of the refinement mask a, the inhomogeneous term g, and the initial vector of functions φ_0.

Key Words: inhomogeneous refinement equation; convergence of cascade algorithm in Sobolev space; transition operator.

1. INTRODUCTION

An inhomogeneous refinement equation is a functional equation of the form

$$\varphi(x)=\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\,\varphi(2x-\alpha)+g(x),\qquad x\in\mathbb{R}^s, \tag{1.1}$$

where $\varphi=(\varphi_1,...,\varphi_r)^T$ is the unknown, $g=(g_1,...,g_r)^T$ is a given vector of functions on $\mathbb{R}^s$, and a is a finitely supported refinement mask such that each $a(\alpha)$ is an $r\times r$ (complex) matrix. Refinement equations play an important role in computer graphics and wavelet analysis. When $g=0$, (1.1) becomes the well-known homogeneous refinement equation

$$\varphi(x)=\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\,\varphi(2x-\alpha),\qquad x\in\mathbb{R}^s. \tag{1.2}$$

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The inhomogeneous refinement equation was used by Strang and Nguyen in [13, p. 294] to investigate boundary scaling functions and wavelets on intervals. Strang and Zhou [14] gave a systematic study of distributional solutions in the univariate case. Distributional solutions of the inhomogeneous refinement equation (1.1) were studied in [3, 9].

To solve the inhomogeneous refinement equation (1.1), we often use the cascade algorithm. If we require a solution φ of compactly supported functions in the Sobolev space $W_2^k(\mathbb{R}^s)$, the inhomogeneous term g must be in $W_2^k(\mathbb{R}^s)$. Starting with an initial vector φ_0 of compactly supported functions in $W_2^k(\mathbb{R}^s)$, the cascade algorithm is defined by

$$\varphi_n(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\, \varphi_{n-1}(2x-\alpha) + g(x), \qquad x \in \mathbb{R}^s, \quad n \in \mathbb{N}. \tag{1.3}$$

We say that the cascade algorithm associated with a, g, and φ_0 is convergent in the Sobolev space $W_2^k(\mathbb{R}^s)$ if there exists an $r \times 1$ vector φ of functions in $W_2^k(\mathbb{R}^s)$ such that

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_{W_2^k(\mathbb{R}^s)} = 0.$$

The convergence of the cascade algorithms is fundamental to wavelet theory and subdivision. Convergence of cascade algorithms has been studied in connection with solutions of refinement equations and the description of curves and surfaces in computer aided geometric design (see [1, 2, 6, 15, 16]). When $s=1$ and $r=1$, a characterization of L_p-convergence of cascade algorithms associated with homogeneous refinement equation (1.2) was given in [2, 6]. The similar characterizations for multivariate and vector homogeneous refinement equations were also respectively given in [5, 11]. In the univariate and scalar case ($s=1$ and $r=1$), Strang and Zhou [14] gave a complete characterization for L_p-convergence of cascade algorithm associated with inhomogeneous refinement equations (1.1). These results were further extended to the multivariate case for $p=2$ [10]. In this paper, we are interested in the strong convergence in the Sobolev space $W_2^k(\mathbb{R}^s)$ of the cascade algorithm associated with inhomogeneous refinement equations (1.1). For the case $r=1$, characterizations of weak and strong convergence in the Sobolev space $W_2^k(\mathbb{R}^s)$ of cascade algorithms associated with homogeneous refinement equation (1.2) were investigated in some papers such as [4, 8]. For the vector case ($r>1$), Micchelli and Sauer [12] investigated convergence in the Sobolev space $W_p^k(\mathbb{R}^s)$ of cascade algorithms associated with homogeneous refinement equations. Our object here is to give a characterization of strong convergence of the cascade algorithm given in (1.3) in the Sobolev space $W_2^k(\mathbb{R}^s)$ in terms of the refinement mask a, the inhomogeneous term g, and the initial vector of functions φ_0.

The main result of this paper provides a sufficient condition for the smoothness of solutions of inhomogeneous refinement equations. Compared with the well-developed smoothness analysis of homogeneously refinable functions (e.g., [16]), the situation for inhomogeneously refinable functions is totally different and their smoothness analysis is not as clear so far.

2. SOME NOTATIONS AND MAIN RESULT

Associated with the refinement equation (1.2), we need the cascade operator Q_a defined by

$$Q_a f(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\, f(2x-\alpha), \qquad f = (f_1, \ldots, f_r)^T \in (L_2(\mathbb{R}^s))^r. \tag{2.1}$$

The Fourier transform of an integrable function f on $\mathbb{R}^s$ is defined to be

$$\hat{f}(\xi) = \int_{\mathbb{R}^s} f(x)\, e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ denotes the inner product of two vectors x and ξ in $\mathbb{R}^s$.

Let $\ell_0(\mathbb{Z}^s)$ (resp. $(\ell_0(\mathbb{Z}^s))^{r\times r}$) denote the linear space of all finitely supported sequences (resp. of $r \times r$ matrices) on $\mathbb{Z}^s$. Furthermore, we denote by $(\ell_\infty(\mathbb{Z}^s))^{r\times r}$ the linear space of all $r \times r$ matrixes of bounded sequences. The norm on $(\ell_\infty(\mathbb{Z}^s))^{r\times r}$ is given by

$$\|v\|_\infty = \sup\{|e_j^T v(\alpha)\, e_l| : \alpha \in \mathbb{Z}^s,\ j, l = 1, \ldots, r\}, \qquad v \in (l_\infty(\mathbb{Z}^s))^{r\times r},$$

where e_j^T denotes the transpose of e_j and e_j is the jth column of $r \times r$ identity matrix.

We use $\mathbb{C}^r$ to denote the linear space of all $r \times 1$ complex vectors. The norm of a vector $\xi = (\xi_1, \ldots, \xi_r)^T \in \mathbb{C}^r$ is defined by $|\xi| = (\sum_{j=1}^r |\xi_j|^2)^{1/2}$.

For a positive integer $k \in \mathbb{N}$, let $W_2^k(\mathbb{R}^s)$ denote the Sobolev space that consists of all vectors of functions f such that $(1+|\xi|)^k \hat{f}(\xi) \in (L_2(\mathbb{R}^s))^r$, equipped with the norm defined by

$$\|f\|_{W_2^k(\mathbb{R}^s)} = \frac{1}{(2\pi)^{s/2}} \left(\int_{\mathbb{R}^s} (1+|\xi|^{2k})\, |\hat{f}(\xi)|^2\, d\xi \right)^{1/2}, \tag{2.2}$$

where $\hat{f}(\xi) = (\hat{f}_1(\xi), \ldots, \hat{f}_r(\xi))^T$ for $f = (f_1, \ldots, f_r)^T$.

Given a finitely supported sequence $c \in (l_0(\mathbb{Z}^s))^{r\times r}$, we use $\tilde{c}(z)$ to denote its symbol

$$\tilde{c}(z) = \sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\, z^\alpha,$$

where $z^\alpha = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ for $z = (z_1, ..., z_s) \in (\mathbb{C}\backslash\{0\})^s$, and $\alpha = (\alpha_1, ..., \alpha_s) \in \mathbb{Z}^s$. For $c, d \in (\ell_0(\mathbb{Z}^s))^{r \times r}$, the discrete convolution of c and d, denoted $c * d$, is given by

$$c * d(\alpha) = \sum_{\beta \in \mathbb{Z}^s} c(\alpha - \beta)\, d(\beta), \qquad \alpha \in \mathbb{Z}^s.$$

It is easily seen that

$$\widetilde{c * d}(z) = \tilde{c}(z)\, \tilde{d}(z), \qquad z = (z_1, ..., z_s) \in (\mathbb{C}\backslash\{0\})^s. \tag{2.3}$$

Let a be an element in $(\ell_0(\mathbb{Z}^s))^{r \times r}$. We define the transition operator F_a to be the linear mapping from $(\ell_0(\mathbb{Z}^s))^{r \times r}$ to $(\ell_0(\mathbb{Z}^s))^{r \times r}$ given by its symbol

$$\widetilde{F_a w}(e^{-i\xi}) = \frac{1}{4^s} \sum_{\lambda \in E} \tilde{a}(e^{-i(1/2)(\xi + 2\pi\lambda)})\, \tilde{w}(e^{-i(1/2)(\xi + 2\pi\lambda)}) \times \tilde{a}(e^{-i(1/2)(\xi + 2\pi\lambda)})^*, \tag{2.4}$$

where $w \in (\ell_0(\mathbb{Z}^s))^{r \times r}$, $\tilde{a}(e^{-i(1/2)(\xi + 2\pi\lambda)})^*$ denotes the complex conjugate transpose of $\tilde{a}(e^{-i(1/2)(\xi + 2\pi\lambda)})$, $E = \{(\varepsilon_1, ..., \varepsilon_s); \varepsilon_1, ..., \varepsilon_s \in \{0, 1\}^s\}$, and $\xi = (\xi_1, ..., \xi_s) \in \mathbb{R}^s$.

Following the ideas of [5, 11], it is easy to show that the minimal invariant subspace W of F_a generated by each $w \in (\ell_0(\mathbb{Z}^s))^{r \times r}$ is finite dimensional. We use $\rho(F_a|_W)$ to denote the spectral radius of $F_a|_W$.

With above notations, our main result is stated as follows.

THEOREM. *Let $k \in \mathbb{N}$, g, and φ_0 be vectors of compactly supported functions in $W_2^k(\mathbb{R}^s)$. Then the cascade algorithm associated with a, g and φ_0 is convergent in the Sobolev space $W_2^k(\mathbb{R}^s)$ if and only if*

$$\lim_{n \to \infty} 2^{2nk} \|F_a^n w_1\|_\infty = 0,$$

and

$$\lim_{n \to \infty} \|F_a^n w_2\|_\infty = 0,$$

where $w_1, w_2 \in (l_0(\mathbb{Z}^s))^{r \times r}$ are given respectively by

$$\tilde{w}_1(e^{-i\xi}) = \sum_{\alpha \in \mathbb{Z}^s} |\xi + 2\pi\alpha|^{2k}\, \hat{g}_0(\xi + 2\pi\alpha)\, \hat{g}_0(\xi + 2\pi\alpha)^*$$

and

$$\tilde{w}_2(e^{-i\xi}) = \sum_{\alpha \in \mathbb{Z}^s} \hat{g}_0(\xi + 2\pi\alpha)\ \hat{g}_0(\xi + 2\pi\alpha)^*,$$

and $g_0 := g + Q_a \varphi_0 - \varphi_0$, *or, equivalently*

$$\rho(F_a|_{W_1}) < 2^{-2k}, \tag{2.5}$$

and

$$\rho(F_a|_{W_2}) < 1, \tag{2.6}$$

where W_1 *and* W_2 *are the minimal invariate subspaces of* F_a *generated respectively by* w_1 *and* w_2.

3. PROOF OF THEOREM

Form (1.3) and (2.1) we have

$$\varphi_n = g + Q_a g + \cdots + Q_a^{n-1} g + Q_a^n \varphi_0.$$

Then

$$\varphi_{n+1} - \varphi_n = Q_a^n g + Q_a^{n+1} \varphi_0 - Q_a^n \varphi_0 = Q_a^n g_0.$$

Hence

$$\|\varphi_{n+1} - \varphi_n\|^2_{W_2^k(\mathbb{R}^s)} = \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} (1 + |\xi|^{2k})\ |\widehat{Q_a^n g_0}(\xi)|^2\ d\xi. \tag{3.1}$$

Let a_n ($n = 1, 2, ...$) be the sequence defined by $a_1 = a$ and

$$a_n(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a_{n-1}(\beta)\ a(\alpha - 2\beta), \qquad \alpha \in \mathbb{Z}^s, \quad n = 2, 3, ... \tag{3.2}$$

It can be easily seen by induction that

$$Q_a^n g_0 = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)\ g_0(2^n \cdot - \alpha).$$

Taking the Fourier transform on both sides, we obtain

$$\widehat{Q_a^n g_0}(\xi) = \frac{1}{2^{ns}}\, \tilde{a}_n(e^{-i2^{-n}\xi})\ \hat{g}_0(2^{-n}\xi), \qquad \xi \in \mathbb{R}^s.$$

It follows that

$$\int_{\mathbb{R}^s} (1+|\xi|^{2k})\, |\widehat{Q_a^n g_0}(\xi)|^2\, d\xi = \frac{1}{4^{ns}} \int_{\mathbb{R}^s} (1+|\xi|^{2k})\, |\tilde{a}_n(e^{-i2^{-n}\xi})\, \hat{g}_0(2^{-n}\xi)|^2\, d\xi$$

$$= \frac{1}{2^{ns}} \int_{\mathbb{R}^s} (1+|2^n\xi|^{2k})\, |\tilde{a}_n(e^{-i\xi})\, \hat{g}_0(\xi)|^2\, d\xi.$$

By periodization, this equals

$$\frac{1}{2^{ns}} \int_{[0,\,2\pi)^s} \sum_{\alpha \in \mathbb{Z}^s} (1+|2^n(\xi+2\pi\alpha)|^{2k})\, |\tilde{a}_n(e^{-i\xi})\, \hat{g}_0(\xi+2\pi\alpha)|^2\, d\xi.$$

By definition of w_1 and w_2 and [7] we obtain

$$2^{ns} \int_{\mathbb{R}^s} (1+|\xi|^{2k})\, |\widehat{Q_a^n g_0}(\xi)|^2\, d\xi = \sum_{j=1}^{r} e_j^T \Bigg[\int_{[0,\,2\pi)^s} \tilde{a}_n(e^{-i\xi})(\tilde{w}_2(e^{-i\xi}) + 2^{2nk}\tilde{w}_1(e^{-i\xi}))\, \tilde{a}_n(e^{-i\xi})^*\, d\xi \Bigg] e_j. \tag{3.3}$$

By the iteration relation (3.2) we have

$$\tilde{a}_n(e^{-i\xi}) = \tilde{a}_{n-1}(e^{-i2\xi})\, \tilde{a}(e^{-i\xi}).$$

Hence

$$\int_{[0,\,2\pi)} \tilde{a}_n(e^{-i\xi})\, \tilde{w}_1(e^{-i\xi})\, \tilde{a}_n(e^{-i\xi})^*\, d\xi$$

$$= \int_{[0,\,2\pi)^s} \tilde{a}_{n-1}(e^{-i2\xi})\, \tilde{a}(e^{-i\xi})\, \tilde{w}_1(e^{-i\xi})\, \tilde{a}(e^{-i\xi})^*\, \tilde{a}_{n-1}(e^{-i2\xi})^*\, d\xi$$

$$= \frac{1}{2^s} \int_{[0,\,4\pi)^s} \tilde{a}_{n-1}(e^{-i\xi})\, \tilde{a}(e^{-i(1/2\,\xi)})\, \tilde{w}_1(e^{-i(1/2)\,\xi}) \times \tilde{a}(e^{-i(1/2)\,\xi})^*\, \tilde{a}_{n-1}(e^{-i\xi})^*\, d\xi.$$

We observe that $[0, 4\pi)^s$ is the disjoint union of $2\pi\lambda + [0, 2\pi)^s$, $\lambda \in E$. Then the above expression equals

$$2^s \int_{[0,\,2\pi)^s} \tilde{a}_{n-1}(e^{-i\xi})\, \widetilde{F_a w_1}(e^{-i\xi})\, \tilde{a}_{n-1}(e^{-i\xi})^*\, d\xi.$$

By induction on n, we obtain

$$\frac{1}{(2\pi)^s}\int_{[0,\,2\pi)^s} \tilde{a}_n(e^{-i\xi})\,\tilde{w}_1(e^{-i\xi})\,\tilde{a}_n(e^{-i\xi})^*\,d\xi = \frac{2^{ns}}{(2\pi)^s}\int_{[0,\,2\pi)^s} \widetilde{F_a^n w_1}(e^{-i\xi})\,d\xi$$
$$= 2^{ns}F_a^n w_1(0). \tag{3.4}$$

Similarly, we have

$$\frac{1}{(2\pi)^s}\int_{[0,\,2\pi)^s} \tilde{a}_n(e^{-i\xi})\,\tilde{w}_2(e^{-i\xi})\,\tilde{a}_n(e^{-i\xi})^*\,d\xi$$
$$= \frac{2^{ns}}{(2\pi)^s}\int_{[0,\,2\pi)^s} \widetilde{F_a^n w_2}(e^{-i\xi})\,d\xi$$
$$= 2^{ns}F_a^n w_2(0). \tag{3.5}$$

From (3.1), (3.3), (3.4), and (3.5), we know that

$$\|\varphi_{n+1}-\varphi_n\|^2_{W_2^k(\mathbb{R}^s)} = \sum_{j=1}^{r} e_j^T(F_a^n w_2(0) + 2^{2nk}F_a^n w_1(0))\,e_j. \tag{3.6}$$

If $\rho(F_a|_{W_1}) < 2^{-2k}$ and $\rho(F_a|_{W_2}) < 1$, then we can find η, $0<\eta<1$, such that $\|F_a^n w_1\|_\infty^{1/n} < \eta 2^{-2k}$ and $\|F_a^n w_2\|_\infty^{1/n} < \eta$ are valid for sufficiently large n. Consequently, there exists a positive constant C independent of n, such that for all $n \in \mathbb{N}$

$$\|F_a^n w_1\|_\infty \leqslant C(2^{-2k}\eta)^n,$$

and

$$\|F_a^n w_2\|_\infty \leqslant C\eta^n.$$

Therefore, we have

$$\|\varphi_{n+1}-\varphi_n\|^2_{W_2^k(\mathbb{R}^s)} \leqslant 2rC\eta^n.$$

Since the supports of φ_n are uniformly bounded, this shows that the sequences $\{\varphi_n\}_{n\in N}$ converge to a vector φ of functions in $W_2^k(\mathbb{R}^s)$. The sufficiency part of theorem is proved.

Next, we establish the necessity part of theorem. From the above discussion, it is easy to obtain

$$F_a^n w_1(\alpha) = \frac{2^{-ns}}{(2\pi)^s}\int_{[0,\,2)^s} \tilde{a}_n(e^{-i\xi})\,\tilde{w}_1(e^{-i\xi})\,\tilde{a}_n(e^{-i\xi})^*\,e^{i\alpha\cdot 2^n\xi}\,d\xi. \tag{3.7}$$

Let $\ell, \tau \in \{1, ..., r\}$. It follows that

$$e_l^T F_a^n w_1(\alpha)\, e_\tau = \frac{2^{-ns}}{(2\pi)^s} \int_{[0,\,2\pi)^s} e_l^T \tilde{a}_n(e^{-i\xi})\, \tilde{w}_1(e^{-i\xi})\, \tilde{a}_n(e^{-i\xi})^*\, e_\tau e^{i\alpha \cdot 2^n \xi}\, d\xi$$

$$= \frac{2^{-ns}}{(2\pi)^s} \int_{[0,\,2\pi)^s} \sum_{\alpha \in \mathbb{Z}^s} |\xi + 2\pi\alpha|^{2k}\, [e_l^T \tilde{a}_n(e^{-i\xi})\, \hat{g}_0(\xi + 2\pi\alpha)]$$

$$\times [e_\tau^T \tilde{a}_n(e^{-i\xi})\, \hat{g}_0(\xi + 2\pi\alpha)]^*\, e^{i\alpha \cdot 2^n \xi}\, d\xi.$$

Hence

$$|e_l^T F_a^n w_1(\alpha)\, e_\tau| \leqslant \frac{2^{-ns}}{(2\pi)^s} \int_{[0,\,2\pi)^s} \sum_{\alpha \in \mathbb{Z}^s} |\xi + 2\pi\alpha|^{2k}\, |e_l^T \tilde{a}_n(e^{-i\xi})\, \hat{g}_0(\xi + 2\pi\alpha)|$$

$$\times |e_\tau^T \tilde{a}_n(e^{-i\xi})\, \hat{g}_0(\xi + 2\pi\alpha)|\, d\xi$$

$$\leqslant \frac{2^{-ns}}{(2\pi)^s} \sum_{j=1}^{r} e_j^T \Bigg[\int_{[0,\,2\pi)^s} \sum_{\alpha \in \mathbb{Z}^s} |\xi + 2\pi\alpha|^{2k}\, \tilde{a}_n(e^{-i\xi})\, \hat{g}_0(\xi + 2\pi\alpha)$$

$$\times \hat{g}_0(\xi + 2\pi\alpha)^*\, \tilde{a}_n(e^{-i\xi})^*\, d\xi \Bigg]\, e_j$$

$$= \frac{2^{-ns}}{(2\pi)^s} \sum_{j=1}^{r} e_j^T \left[\int_{[0,\,2\pi)^s} \tilde{a}_n(e^{-i\xi})\, \tilde{w}_1(e^{-i\xi})\, \tilde{a}_n(e^{-i\xi})^*\, d\xi \right] e_j$$

$$= \sum_{j=1}^{r} e_j^T F_a^n w_1(0)\, e_j. \tag{3.8}$$

Similarly, we obtain

$$|e_l^T F_a^n w_2(\alpha)\, e_\tau| \leqslant \sum_{j=1}^{r} e_j^T F_a^n w_2(0)\, e_j. \tag{3.9}$$

If one of (2.5) and (2.6) does not hold, then we have

$$\inf_{n \geqslant 1} \|F_a^n|_{W_1}\|_\infty^{1/n} = \lim_{n \to \infty} \|F_a^n|_{W_1}\|_\infty^{1/n} \geqslant 2^{-2k},$$

or

$$\inf_{n \geqslant 1} \|F_a^n|_{W_2}\|_\infty^{1/n} = \lim_{n \to \infty} \|F_a^n|_{W_2}\|_\infty^{1/n} \geqslant 1.$$

It follows that

$$\|F_a^n|_{W_2}\|_\infty \geqslant 1, \qquad n \in \mathbb{N},$$

or

$$2^{2nk}\, \|F_a^n|_{W_1}\|_\infty \geqslant 1, \qquad n \in \mathbb{N}.$$

From the proof of Lemma 2.4 in [5], we see that there exists a positive constant M independent of n such that

$$\|F_a^n w_2\|_\infty \geqslant M, \qquad n \in \mathbb{N},$$

or

$$2^{2nk}\, \|F_a^n w_1\|_\infty \geqslant M, \qquad n \in \mathbb{N}.$$

This in connection with (3.6), (3.8), and (3.9) gives

$$\|\varphi_{n+1} - \varphi_n\|_{W_2^k(\mathbb{R}^s)} \geqslant M, \qquad n \in \mathbb{N}.$$

This contradicts the fact that

$$\lim_{n\to\infty} \|\varphi_{n+1} - \varphi_n\|_{W_2^k(\mathbb{R}^s)} = 0.$$

The necessity part of the theorem is also proved.

4. EXAMPLE

In this section we give an example considered in [10, 14] to illustrate our theory.

EXAMPLE. Consider the inhomogeneous refinement equation

$$\varphi(x) = t\varphi(2x) + t\varphi(2x-1) + g(x), \qquad x \in \mathbb{R}, \tag{4.1}$$

where t is a nonzero complex number and g is a function in $W_2^k(\mathbb{R}^s)$ supported in $[0, 2]$. Let φ_0 be a function in $W_2^k(\mathbb{R}^s)$ supported in $[0, 2]$. The corresponding cascade algorithm is given by

$$\varphi_{n+1}(x) = t\varphi_n(2x) + t\varphi_n(2x-1) + g(x), \qquad n = 1, 2, \ldots .$$

We have $a(0) = t$, $a(1) = t$ and $a(\alpha) = 0$ for $\alpha \notin \{0, 1\}$. Let b be the sequences given by $b = (a * a^*)/2$, where $a^*(\alpha) = \overline{a(-\alpha)}$, $\alpha \in \mathbb{Z}$. Then $b(-1) = |t|^2/2$, $b(0) = |t|^2$, $b(1) = |t|^2/2$, and $b(\alpha) = 0$ for $\alpha \notin \{-1, 0, 1\}$. Let $g_0(x) = g(x) + t\varphi_0(2x) + t\varphi_0(2x-1) - \varphi_0(x)$. Combining Example 3.2 in [10] and

Section 5 in [14] with our theorem, we know that if $|t| < 2^{-k}$, the cascade algorithm associated with a, g, and any φ_0 is convergent in $W_2^k(\mathbb{R}^s)$; if $2^{-k} \leqslant |t| < 2^{-k+(1/2)}$, the cascade algorithm associated with a, g, and φ_0 is convergent in $W_2^k(\mathbb{R}^s)$ if and only if

$$\int_{\mathbb{R}} |g_0^{(k)}(x)|^2 \, dx + \int_{\mathbb{R}} g_0^{(k)}(x)(\overline{g_0^{(k)}(x+1)} + \overline{g_0^{(k)}(x-1)}) \, dx = 0;$$

if $2^{-k+(1/2)} \leqslant |t| < 1$, the cascade algorithm associated with a, g, and φ_0 is convergent in $W_2^k(\mathbb{R}^s)$ if and only if $g_0^{(k)}(x) = 0$; if $1 \leqslant |t| < 2^{1/2}$, the cascade algorithm associated with a, g, and φ_0 is convergent in $W_2^k(\mathbb{R}^s)$ if and only if $g_0^{(k)}(x) = 0$ and

$$\int_{\mathbb{R}} |g_0(x)|^2 \, dx + \int_{\mathbb{R}} g_0(x)(\overline{g_0(x+1)} + \overline{g_0(x-1)}) \, dx = 0;$$

and if $|t| \geqslant 2^{1/2}$, the cascade algorithm associated with a, g, and φ_0 is convergent in $W_2^k(\mathbb{R}^s)$ if and only if φ_0 is a solution of Eq. (4.1).

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